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Statistical field theories deformed within different calculi

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Abstract. Within the framework of basic-deformed and finite-difference calculi, as well as deformation procedures proposed by Tsallis, Abe, and Kaniadakis and generalized by Naudts, we develop field-theoretical schemes of statistically distributed fields. We construct a set of generating functionals and find their connection with corresponding correlators for basic-deformed, finite-difference, and Kaniadakis calculi. Moreover, we introduce pair of additive functionals, which expansions into deformed series yield both Green functions and their irreducible proper vertices. We find as well formal equations, governing by the generating functionals of systems which possess a symmetry with respect to a field variation and are subjected to an arbitrary constrain. Finally, we generalize field-theoretical schemes inherent in concrete calculi in the Naudts manner. From the physical point of view, we study dependences of both one-site partition function and variance of free fields on deformations. We show that within the basic-deformed statistics dependence of the specific partition function on deformation has in logarithmic axes symmetrical form with respect to maximum related to deformation absence; in case of the finite-difference statistics, the partition function takes non-deformed value; for the Kaniadakis statistics, curves of related dependences have convex symmetrical form at small curvatures of the effective action and concave form at large ones. We demonstrate that only moment of the second order of free fields takes non-zero values to be proportional to inverse curvature of effective action. In dependence of the deformation parameter, the free field variance has linearly arising form for the basic-deformed distribution and increases non-linearly rapidly in case of the finite-difference statistics; for more complicated case of the Kaniadakis distribution, related dependence has double-well form.

1 Introduction

In the course of the complex system investigations, great variety of statistical theories has been developed [1–11]. The approaches related are based on the principal peculiarity of the statistical behaviour of complex systems which is known to be their complicated dynamics being spanned in the fractal phase space governed by long-range interaction or long-time memory effects [12–15]. These complications have been overcome within framework of the standard statistical approach [16] by means of modification of the Boltzmann-Gibbs distribution due to deformations of the exponential function. Formally, the approaches proposed are based on using the extremum principle for a deformed entropic forms where the ordinary logarithm function is substituted with some versions modified according to Tsallis [1], Abe [4], Kaniadakis [5], Naudts [6], basic deformation procedure [11], etc. To the best of our knowledge, the only effort along of direct using the field-theoretical method for development of a statistical scheme has been attempted in the work [17] for the Tsallis thermostatistics. The present article is created with the purpose to generalize the standard statistical field theory [18,19] within different versions of calculi elaborated to this moment.

Historically the first example of such calculus gives the basic-deformed calculus (q-calculus, in other words) which has been originally introduced by Heine and Jackson [20–23] in the study of basic hypergeometric series [24,25]. It appears the q-calculus does not only play a central role in the quantum groups and algebra belonging to mathematical branches, but have a deep physical meaning [26,27]. In this context, study of q-deformed bosons and fermions shows that thermodynamics can be built on the formalism of q-calculus where the ordinary derivative is substituted by use of appropriate Jackson derivative and q-integral [28]. Quantum algebra has subsequently found several applications in different physical fields, such as cosmic strings and black holes [29], conformal quantum mechanics [30], nuclear and high energy physics [31,32], fractional quantum Hall effect and high-T_c superconductors [33]. Deformations of exponential function have been proposed in the context

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of non-extensive statistic mechanics [1–4,15,34,35], relativistic statistical theory [5,36], econophysics [37]. The areas, where deformations of the exponential functions have been treated, include both formal mathematical and theoretical physical developments [1,5,6,15,29,30,36,38,39] and observation of consistent concordance with experimental behaviour [32,36,37]. Moreover, being based on a scale transformation related to the Jackson q-derivative and q-integral, the basic calculus is appropriate to describe multifractal sets [40,41]. Displaying critical phenomena of the type of growth processes, rupture, earthquake, financial crashes, these systems reveal a discrete scale invariance with the existence of log-periodic oscillations deriving from a partial breakdown of the continuous scale invariance symmetry into a discrete one – as occurs, for example, in hierarchical lattices [42–47].

Concerning the formalism on which our method is based, it is necessary to highlight that it reduces to use of generating functional which presents the Fourier-Laplace transform of the partition function from the dependence on the fluctuating distribution of an order parameter to an auxiliary field [18,19]. Due to the exponential character of this transform, determination of correlators of the order parameter is provided by differentiation of the generating functional over auxiliary field. As it was mentioned above, fractality of the phase space of complex systems causes deformation of both exponential function and integral itself in the Fourier-Laplace transform, so that the simple procedure of differentiation of the generating functional becomes inconsistent. Thus, main problem of our consideration reduces to finding derivation operators, which action keeps form of deformed exponentials giving eigen numbers related (within the basic calculus, such derivative reduces to the Jackson one, while eigen number is presented by basic-number [48]).

This paper is devoted to development of the field-theoretical scheme basing on possible generalizations of both Fourier-Laplace transform and derivative operator related. The work is organized as follows. In Section 2 we yield a necessary information from the theory of both basic calculus and finite-difference calculus (h-calculus); moreover, we consider as well main peculiarities of the deformation procedures according to Tsallis, Abe, Kaniadakis, and Naudts. Sections 3–5 are devoted to construction of the generating functionals and finding their connection with related correlators for basic-deformed, finite-difference, and Kaniadakis calculi. Within the simplest harmonic approach, we find the partition functions and the order parameter moments as functions of the deformation parameters. Moreover, we introduce pair of additive functionals, which expansions into deformed series yield both Green functions and their proper vertices; we also find formal equations governed by the generating functionals of systems, which possess a symmetry with respect to a field variation and are subjected to an arbitrary constrain. In Section 6 we generalize the field-theoretical schemes elaborated in previous Sections. Section 7 concludes our consideration and Appendix A contains calculations related to deformed Gamma functions.

2 Preliminaries

Before stating main properties of the basic-deformed, the finite-difference, the Tsallis, the Abe, the Kaniadakis, and the Naudts-type calculi, we should stress the principal difference between the basic and algebraic deformations (the first two of mentioned calculi relate to the basic deformations, while the rest – to the algebraic ones). The basic-deformed formalism consists in a non-trivial deformation of the algebraic structure, when such fundamental notions as the number (transforming into q- and h-deformed numbers), the derivative, the integral, and so on are changed. The algebraic deformation formalisms keep the algebraic structure in the standard form (for example, real and complex numbers do not change their nature), but generalized operations are introduced (deformed sum, product, Fourier-transform, etc.). In both cases deformed functions are introduced to denote composition of elementary functions forming fundamental elements of the formalism related. Typically, in the case of basic deformation, functions that keep partially their usual properties can be found: for example, every basic-deformed exponential has a dual counterpart for which the multiplication rule is obeyed; inverse functions related are basic-deformed logarithm and its dual counterpart which obey the addition rule. It is not the case for algebraically deformed functions.

1. We start by referring to the standard definition of the dual pair of *the basic exponentials* in form of the Taylor series [48]

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad E_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} q^{\frac{n(n-1)}{2}}. \tag{1}$$

These series are seen to be determined by the factorials

$$[n]_q! = [1]_q \cdot [2]_q \cdot \dots \cdot [n-1]_q \cdot [n]_q \tag{2}$$

of the basic numbers

$$[n]_q := \frac{q^n - 1}{q - 1} \tag{3}$$

with the limit $[n]_{q\to 1} = n$ due to which the functions (1) transform into the ordinary exponential at $q \to 1$. Since $[n]_{1/q} = [n]_q q^{-(n-1)}$, one has the relation $[n]_{1/q}! = [n]_q! q^{-\frac{n(n-1)}{2}}$ which yields the connection

$$E_q(x) = e_{1/q}(x). \tag{4}$$

The mutual complementarity of the basic exponentials (1) is stipulated from the multiplication rule [48]

$$E_q(x) e_q(y) = e_q(x + y) \tag{5}$$

according to which

$$E_q(x) e_q(-x) = 1. \tag{6}$$

It is natural to introduce pair of the basic logarithmic functions $\mathrm{ln}_q(x)$ and $\mathrm{Ln}_q(x)$ being inverse to the exponentials (1): $\mathrm{ln}_q\big(e_q(x)\big) = x$ and $\mathrm{Ln}_q\big(E_q(x)\big) = x$. If the operation $\ln_q$ is applied to the equation (5), the equality $x+y = \ln_q\big(E_q(x)e_q(y)\big)$ is obtained which becomes the identity under condition that pair of dual q-logarithms is obeyed the addition rule

$$\ln_q(xy) = \ln_q(x) + \mathrm{Ln}_q(y). \tag{7}$$

The principal peculiarity of the exponentials (1) is to keep their forms under action of *the Jackson derivative*

$$\mathcal{D}^q_x f(x) := \frac{f(qx) - f(x)}{(q-1)x}. \tag{8}$$

In fact, one has for arbitrary constants a and b [48]

$$\begin{aligned} \mathcal{D}^q_x e_q(ax+b) &= a e_q(ax+b), \\ \mathcal{D}^q_x E_q(ax+b) &= a E_q(qax+b). \end{aligned} \tag{9}$$

In this way, for arbitrary functions $f(x)$ and $g(x)$ the Leibnitz rule reads:

$$\begin{aligned} \mathcal{D}^q_x\left[f(x)g(x)\right] &= g(qx)\mathcal{D}^q_x f(x) + f(x)\mathcal{D}^q_x g(x) \\ &= g(x)\mathcal{D}^q_x f(x) + f(qx)\mathcal{D}^q_x g(x). \end{aligned} \tag{10}$$

Let us write as well some useful equations

$$\begin{aligned} e_q(q^n x) &= \Big(1 + (q-1)\,x\Big)^n_q e_q(x), \\ E_q(q^n x) &= \frac{E_q(x)}{\Big(1 - (q-1)\,x\Big)^n_q} \end{aligned} \tag{11}$$

following from the definition (8) and the equalities (9) if $a=1$ and $b=0$. Here we use the basic-deformed binomial [48]

$$(x+y)^n_q := (x+y)(x+qy)\dots(x+q^{n-1}y). \tag{12}$$

In addition to the basic exponentials (1) being invariant with respect to action of the Jackson derivative (8), its eigenfunctions are known to represent the homogeneous functions determined with the property

$$h(qx) = q^\alpha h(x) \tag{13}$$

where an exponent α plays the role of the self-similarity degree, q is an arbitrary factor playing the role of deformation of self-similar systems for which the homogeneous functions are the basis of the statistical theory related [47]. To this end, the eigenvalues of the Jackson derivative (8) determined on the set of the homogeneous functions represent the basic numbers (3):

$$\mathcal{D}^q_x h(x) = [\alpha]_q h(x), \quad [\alpha]_q = \frac{q^\alpha - 1}{q-1}. \tag{14}$$

Apart from invariant action of the Jackson derivative $\mathcal{D}^q_x$, the exponentials (1) are persistent as well under action of *the basic integral* $\mathcal{I}^q_x$ defined as [48]

$$\mathcal{I}^q_x f(x) \equiv \int f(x) d_q x := (1-q)x \sum_{n=0}^{\infty} f(q^n a) q^n. \tag{15}$$

Meaning $\mathcal{D}^q_x$ and $\mathcal{I}^q_x$ as transformation operators of the Lee group, let us introduce the generators related:

$$\begin{aligned} u^q_x &:= \ln_q(\mathcal{D}^q_x), & j^q_x &:= \ln_q(\mathcal{I}^q_x), \\ U^q_x &:= \mathrm{Ln}_q(\mathcal{D}^q_x), & J^q_x &:= \mathrm{Ln}_q(\mathcal{I}^q_x). \end{aligned} \tag{16}$$

Applying the dual basic-logarithmic functions to the formal relation $\mathcal{D}^q_x\mathcal{I}^q_x = 1$, with use of the equality $\ln_q(1) = 0$ and the property (7) the equations $u^q_x + J^q_x = 0$, $U^q_x + j^q_x = 0$ are obtained. These equations yield the functional expressions of the basic integral through the Jackson derivative:

$$\mathcal{I}^q_x = e_q\big(-\mathrm{Ln}_q\left(\mathcal{D}^q_x\right)\big) = E_q\big(-\ln_q\left(\mathcal{D}^q_x\right)\big). \tag{17}$$

Further we need the improper integral to be defined by the expressions [48]

$$\begin{aligned} \int_0^\infty f(x) d_q x &= \sum_{n=-\infty}^{n=\infty} \int_{q^{n+1}}^{q^n} f(x) d_q x, \\ \int_0^\infty f(x) d_q x &= \sum_{n=-\infty}^{n=\infty} \int_{q^n}^{q^{n+1}} f(x) d_q x \end{aligned} \tag{18}$$

the first of which relates to the deformation parameter $0 < q < 1$, the second – to $q > 1$.

2. Let us list now main cases of deformations that complete the basic deformation related to the exponentials (1). We have demonstrated above the dual pair of these exponentials is determined by the numbers $[n]_q$ and $[n]_{1/q}$ defined by equation (3). In the case of *the symmetrized q-calculus*, the corresponding number

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \tag{19}$$

fulfils the condition $[n]_{1/q} = [n]_q$, so that the dual exponential $E_q(x) = e_{1/q}(x)$ coincides with the original one, $e_q(x)$. Starting from the $q \leftrightarrow 1/q$ symmetry, Abe has founded the entropy and the related statistics being inherent in the symmetrized basic-calculus [4].

3. The next example gives *the h-exponential* that is specified by the expression [48]

$$e_h(x) := (1+h)^{\frac{x}{h}}. \tag{20}$$

This function is inverse to the h-logarithm

$$\ln_h(x) = \frac{h\ln(x)}{\ln(1+h)} \tag{21}$$

with $\ln(x)$ being the ordinary logarithm function. It is easy to make sure that the expression (20) may be presented in form of the series (1) if the basic numbers (3) are substituted by the h-numbers

$$[n]_h := \frac{hn}{\ln(1+h)}. \tag{22}$$

It is enough for our aims to consider the positive h values only. Then it is natural to suppose the symmetry with respect to change of the h sign. Defining the self-dual h-exponential as $E_h(x) := e_{-h}(x) = e_h(x)$, we obtain the trivial rule

$$e_h(x+y) = e_h(x)e_h(y) \tag{23}$$

instead of equation (5). In the limit $h \to 0$, the h-calculus reduces naturally to the usual case.

The h-exponential (20) is obviously invariant with respect to action of the h-derivative [48]

$$D_x^h f(x) := \frac{f(x+h) - f(x)}{h}. \tag{24}$$

However, the properties (9) take the form

$$\begin{aligned} D_x^h e_h(ax+b) &= d_h(a) e_h(ax+b), \\ d_h(a) &\equiv \frac{e_h(ah) - 1}{h} = \frac{(1+h)^a - 1}{h} \end{aligned} \tag{25}$$

complicated with the factor $d_h(a)$. Although this factor has the ordinary limit $d_{h\to 0}(a) \to a$, action of the h-derivative (24) on the exponential (20) does not fulfil a condition of the type (9) for arbitrary values of a constant a. To restore such a condition we shall use the definition of the h-derivative

$$\mathcal{D}_x^h := [1]_h \partial_x, \quad [1]_h = \frac{h}{\ln(1+h)}, \; \partial_x \equiv \frac{\partial}{\partial x} \tag{26}$$

instead of equation (24). Being applied to the exponential (20), this derivative ensures obviously the first property (9) with index q substituted by h. Respectively, the h-integral, being inverse to the derivative (26), is defined as

$$\mathcal{I}_x^h f(x) = \int f(x) d_h x, \quad d_h x = [1]_h^{-1} dx. \tag{27}$$

4. Quite different example represents *the Tsallis exponential*

$$\exp_q(x) := [1 + (1-q)x]^{\frac{1}{1-q}} \tag{28}$$

characterized by the deformed number

$$\{n\}_q = \frac{n}{1 + (1-q)(n-1)}. \tag{29}$$

Here, the dual number $\{n\}_{1/q}$ relates to the exponential

$$\mathrm{Exp}_q(x) := \exp_{1/q}(x) = \left[\exp_q(-x/q)\right]^{-q}, \tag{30}$$

which inverse value is related to the escort probability being the basis of the Tsallis thermostatistics [3,15]. For our aims the following is principally important: (i) the Tsallis exponential (28) is not invariant with respect to the Jackson derivative (8), whereas action of the ordinary derivative gives $\frac{d}{dx}\exp_q(x) = \exp_q^q(x)$; (ii) the dual exponential (30) fulfils the condition $\mathrm{Exp}_q^{1/q}(-qx)\exp_q(x) = 1$ that does not coincide with the condition (6); (iii) the Tsallis exponential (28) is invariant with respect to the derivative $[1+(1-q)x]\frac{d}{dx}$ [35]. In the limit $q \to 1$, the Tsallis calculus reduces to the ordinary one. Formally, the Tsallis exponential coincides with the h-one at arbitrary values of the deformation parameter $q = 1 - h/x$. However, the Tsallis calculus does not transform into h-calculus, because the former requires to be constant the parameter q, while the latter makes the same for h. Due to this, the Tsallis exponential increases with the x-growth according to the power law (28), while the h-exponential (20) varies exponentially. The statistical field theory based on the Tsallis calculus has been developed in the work [17].

5. As the following example, we consider the case of *the Kaniadakis deformation* when exponential and logarithm functions are defined as follows [5]:

$$\exp_\kappa(x) := \left[\kappa x + \sqrt{1 + (\kappa x)^2}\right]^{1/\kappa}, \tag{31}$$

$$\ln_\kappa(x) := \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \tag{32}$$

Here, the deformation parameter κ belongs to the interval $(-1, 1)$ and the limit $\kappa \to 0$ relates to the ordinary functions $\exp(x)$ and $\ln(x)$. The exponential (31) is self-dual function in the sense that it fulfils the condition $\exp_\kappa(x)\exp_\kappa(-x) = 1$ of the type (6). However, the multiplication rule (5) takes the form [49]

$$\exp_\kappa(x)\exp_\kappa(y) = \exp_\kappa\left(x \overset{\kappa}{\oplus} y\right) \tag{33}$$

where the sum is deformed as

$$x \overset{\kappa}{\oplus} y := x\sqrt{1 + (\kappa y)^2} + y\sqrt{1 + (\kappa x)^2}. \tag{34}$$

It is worthy to note that the rule of the type (33) takes place also for the Tsallis calculus where the deformed sum (34) is written as [35]

$$x \oplus_q y := x + y + (1-q)xy. \tag{35}$$

The κ-exponential (31) can be written formally as the Tailor series

$$\exp_\kappa(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!_\kappa} \tag{36}$$

with deformed factorial

$$\begin{aligned} n!_\kappa &= \frac{n!}{\nu_\kappa(n)}; \quad \nu_\kappa(0) = \nu_\kappa(1) = 1, \\ \nu_\kappa(n) &= \prod_{m=1}^{n-1} \left[1 - (2m-n)\kappa\right], \; n > 1. \end{aligned} \tag{37}$$

Since the factor $\nu_\kappa(n)$ depends on the number n, the κ-deformed factorial (37) can not be presented in the form of the factorial (2). However, it is easy to convince that the exponential (31) is invariant with respect to action of the derivation operator [49]

$$\mathcal{D}_x^\kappa \equiv \frac{\partial}{\partial_\kappa x} := \sqrt{1 + (\kappa x)^2}\, \frac{\partial}{\partial x}. \tag{38}$$

Moreover, for arbitrary constant a one obtains

$$\mathcal{D}_x^{a\kappa} \exp_\kappa(ax) = a \exp_\kappa(ax) \tag{39}$$

instead of equations (9) and (25). The integration operator being inverse to the derivative (38) is defined in the relativistic form [49]

$$\mathcal{I}_x^\kappa f(x) := \int f(x) d_\kappa x, \quad d_\kappa x \equiv \frac{dx}{\sqrt{1+(\kappa x)^2}}. \tag{40}$$

The derivative property (39) accompanied with the integral definition (40) will be shown to be formal statement for development of the field-theoretical scheme based on the Kaniadakis calculus.

It is worthy to stress that the logarithm (32) can be generalized to the form [50]

$$\ln_{\kappa\tau\varsigma}(x) := \frac{x^\tau\left[(\varsigma x)^\kappa - (\varsigma x)^{-\kappa}\right] - (\varsigma^\kappa - \varsigma^{-\kappa})}{(\kappa+\tau)\varsigma^\kappa + (\kappa-\tau)\varsigma^{-\kappa}} \tag{41}$$

being solution of the functional equation

$$\partial_x\left[x\Lambda(x)\right] = \lambda\Lambda\left(x/\alpha\right) + \eta \tag{42}$$

with parameters

$$\begin{aligned} \alpha &= \left(\frac{1+\tau-\kappa}{1+\tau+\kappa}\right)^{\frac{1}{2\kappa}}, \\ \lambda &= \frac{(1+\tau-\kappa)^{\frac{\tau+\kappa}{2\kappa}}}{(1+\tau+\kappa)^{\frac{\tau-\kappa}{2\kappa}}}, \\ \eta &= (\lambda-1)\frac{\varsigma^\kappa - \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^\kappa + (\kappa-\tau)\varsigma^{-\kappa}}. \end{aligned} \tag{43}$$

The generalized logarithm (41) yields known cases of the following deformations: (i) the choice of parameters $\tau = 0$, $\varsigma = 1$, and $\kappa \in (-1, 1)$ relates to the Kaniadakis deformation [5]; (ii) $\kappa = -\tau = (1-q)/2$ – the Tsallis deformation with parameter q [1]; (iii) $\kappa = (q-1/q)/2$, $\tau = (q+1/q)/2$, and $\varsigma = 1$ – the Abe deformation with parameter q [4]; (iv) the choice $\varsigma = 1$ relates to the two-parameter logarithm proposed by Mittal and Sharma [51,52]; (v) the case $\tau = 0$ relates to the scaled two-parameter logarithm proposed by Kaniadakis [50].

6. Finally, we note one more example of generalized logarithms – the Naudts functionally deformed logarithm [6]

$$\ln_\phi(x) := \int_1^x \frac{dx'}{\phi(x')} \tag{44}$$

specified with a function $\phi(x)$. As usually, corresponding exponential function $\mathrm{e}_\phi(x)$ is defined by the condition $\mathrm{e}_\phi\left[\ln_\phi(x)\right] = x$. Formally, we may as well define a derivation operator D_x^ϕ which keeps the form of the functionally deformed exponential according to the equation $\mathrm{D}_x^\phi \mathrm{e}_\phi(x) = \eta_\phi \mathrm{e}_\phi(x)$ with an eigenvalue η_ϕ fixed by the $\phi(x)$ function choice.

Examples considered above show that a generalized calculus can be built as a result of the following steps:

– Choose a deformed exponential $\mathrm{e}_\lambda(x)$ and find its dual form $\mathrm{E}_\lambda(x)$ obeying the multiplication rule

$$\mathrm{E}_\lambda(x)\mathrm{e}_\lambda(y) = \mathrm{e}_\lambda(x+y) \tag{45}$$

with λ being a generalized deformation parameter. If above exponentials may be expanded into the Taylor series of the type (1), their choice is fixed by numbers $[n]_\lambda$ generalizing the expressions (3), (19), and (22) with parameter q being substituted with a deformation λ. In the case of the type of both Tsallis and Kaniadakis deformations, more convenient to use a self-dual exponential obeying the multiplication rule

$$\mathrm{e}_\lambda(x)\mathrm{e}_\lambda(y) = \mathrm{e}_\lambda\left(x \overset{\lambda}{\oplus} y\right) \tag{46}$$

defined by specifying deformed sum $x \overset{\lambda}{\oplus} y$ (see, for example, Eqs. (34) and (35)).

– Define a deformed differentiation operator D_x^λ type of the Jackson derivative (8) according to the condition that this operator keeps invariant forms of the generalized exponentials $\mathrm{e}_\lambda(x)$ and $\mathrm{E}_\lambda(x)$.

– Introduce a deformed integration operator according to the definitions

$$\mathrm{I}_x^\lambda = \mathrm{e}_\lambda\left[-\mathrm{Ln}_\lambda\left(\mathrm{D}_x^\lambda\right)\right] = \mathrm{E}_\lambda\left[-\ln_\lambda\left(\mathrm{D}_x^\lambda\right)\right] \tag{47}$$

generalizing equations (17) (here, deformed logarithmic functions $\ln_\lambda(x)$ and $\mathrm{Ln}_\lambda(x)$ are defined to be inverse to the generalized exponentials related).

As a result, we achieve the position to develop a statistical field theory that is based on the use of a generating functional being a generalization of the characteristic function [18,19]. This function is known to be presented by the Fourier-Laplace transform

$$p(j) := \mathrm{I}_x^\lambda\left[p(x)\mathrm{E}_\lambda(jx)\right] = \int p(x)\mathrm{E}_\lambda(jx) d_\lambda x \tag{48}$$

of the probability distribution $p(x)$. The key point is that deformed exponential standing within integrand of the characteristic function (48) is eigenfunction of the deformed derivative operator D_j^λ with an eigenvalue $d_\lambda(x)$. As a result, multiple differentiation of this function over auxiliary variable j keeps its exponential form to yield the moments

$$\begin{aligned} \langle [d_\lambda(x)]^n \rangle_\lambda &:= \int [d_\lambda(x)]^n p(x) d_\lambda x \\ &= \left(\mathrm{D}_j^\lambda\right)^n p(j)\Big|_{j=0} \end{aligned} \tag{49}$$

of an order parameter $d_\lambda(x)$.

3 Basic-deformed statistics

Let us consider a statistical system, which distribution over states $\mathbf{x} = \{\mathbf{r}_a, \mathbf{p}_a\}$ in the phase space of particles

$a = 1, \ldots, N$, $N \to \infty$ with coordinates $\mathbf{r}_a$ and momenta $\mathbf{p}_a$ is determined by a Hamiltonian $H = H(\mathbf{x})$. We are interested in study of the coarse space distribution $\phi(\mathbf{r})$ of an order parameter ϕ. Within the coarse grain approximation, thermostatistics of the deformed system is governed by the partition functional

$$\mathcal{Z}_q\{\phi\} := \int e_q\left[-\beta H(\mathbf{x})\right]\delta\left[\phi - \phi(\mathbf{x})\right] d_q\mathbf{x} \equiv e_q\left(-S\{\phi\}\right). \tag{50}$$

Here, $d_q\mathbf{x} = (q-1)\mathbf{x}$ stands for the basic-deformed differential, $S = S\{\phi\}$ is an effective action, and we take into account also that thermostatistical distribution of a basic-deformed system is proportional to the q-deformed exponential $e_q\left[-\beta H(\mathbf{x})\right]$ with the inverse temperature β measured in the energy units [11].

The principle peculiarity of the definition (50) is that both thermostatistical exponential and integral over the phase space are the basic deformed ones. To this end, we need to use the basic-deformed Laplace transform

$$\mathcal{Z}_q\{J\} := \int \mathcal{Z}_q\{\phi\} E_q\{J\cdot\phi\}\{d_q\phi\} = \int e_q\left(-S\{\phi\} + J\cdot\phi\right)\{d_q\phi\} \tag{51}$$

where the last equation is written with consideration of the property (5). For the sake of simplicity, we use the lattice representation to describe the coordinate dependence by means of the index $i = 1, \ldots, N$ in the shorthands $J\cdot\phi \equiv \sum_i J_i\phi_i$ and $\{d_q\phi\} \equiv \prod_i d_q\phi_i$.

According to the rules (9) the n-fold differentiation of the last expression for the generating functional (51) yields

$$\left(\mathcal{D}^q_{J_1}\ldots\mathcal{D}^q_{J_n}\right)\mathcal{Z}_q\{J\} = \int \left(\phi_{i_1}\ldots\phi_{i_n}\right) e_q \times \left(-S\{\phi\} + J\cdot\phi\right)\{d_q\phi\}. \tag{52}$$

The right hand side of this equality determines the correlator

$$\left\langle \phi_{i_1}\ldots\phi_{i_n}\right\rangle_q := \mathcal{Z}_q^{-1}\int \left(\phi_{i_1}\ldots\phi_{i_n}\right)\mathcal{Z}_q\{\phi\}\{d_q\phi\} \tag{53}$$

where the coefficient is inversely proportional to the partition function

$$\mathcal{Z}_q := \int \mathcal{Z}_q\{\phi\}\{d_q\phi\} = \int e_q\left(-S\{\phi\}\right)\{d_q\phi\}. \tag{54}$$

Combination of the last equalities allows to express an arbitrary correlator through the basic-deformed derivatives of the generating functional (51):

$$\left\langle \phi_{i_1}\ldots\phi_{i_n}\right\rangle_q = \mathcal{Z}_q^{-1}\left(\mathcal{D}^q_{J_1}\ldots\mathcal{D}^q_{J_n}\right)\mathcal{Z}_q\{J\}\Big|_{J_1,\ldots,J_n=0}. \tag{55}$$

Within the framework of the harmonic approach, effective action takes the deformed parabolic form

$$S^{(0)}\{\phi\} = \sum_i^N S^{(0)}\{\phi_i\}, \quad S^{(0)}\{\phi_i\} \equiv \frac{(\phi_i)_q^2}{[2]_q \varDelta^2} \tag{56}$$

where the deformed binomial (12) is used, $[2]_q = 1 + q$, and $\varDelta^2$ stands for the inverse curvature. To apply the rule (5) for the basic-deformed exponential in the generating functional (51) let us suppose the symmetry with respect to substituting the deformation parameter q by the inverse value $1/q$. Then, it is convenient to separate whole lattice into odd sites i' and even ones i'' and use equation (5) for each of couples i', i''. To this end, the exponential in the partition function (54) is transformed as follows:

$$e_q\left(-S^{(0)}\{\phi\}\right) = e_q\left(\sum_i^N S^{(0)}\{\phi_i\}\right) = \prod_{i'}^{[N/2]} e_q\left(S^{(0)}\{\phi_{i'}\}\right) \times \prod_{i''}^{[N/2]} e_{1/q}\left(S^{(0)}\{\phi_{i''}\}\right) \tag{57}$$

where square brackets denote the integer of the fraction $N/2$.[1] As a result, the generating functional (51) takes the multiplicative form

$$\mathcal{Z}_q^{(0)}\{J\} = \prod_{i'}^{[N/2]} z_q^{(0)}(J_{i'}) \prod_{i''}^{[N/2]} z_{1/q}^{(0)}(J_{i''}). \tag{58}$$

As simple calculations in Appendix A show, each of multipliers related to one site is determined by the expression

$$z_q^{(0)}(J) = \frac{2\varDelta}{\sqrt{[2]_q}}\gamma_q\left(\frac{1}{2}\right) E_q\left[\frac{q}{[2]_q}(\varDelta J)^2\right] \tag{59}$$

where the basic-deformed γ-function is defined by the first equation (A.1). Respectively, specific partition function $z_q^{(0)} \equiv z_q^{(0)}(J=0)$ reads:

$$z_q^{(0)} = \frac{2\varDelta}{\sqrt{[2]_q}}\gamma_q\left(\frac{1}{2}\right). \tag{60}$$

As Figure 1a shows, the dependence of this function on the deformation parameter has in logarithmic axes symmetrical form with respect to the maximum point $q = 1$.

According to the definition (55), within the harmonic approach the order parameter is determined as

$$\langle\phi\rangle_q^{(0)} = \frac{q}{[2]_q}\frac{(1+q) - q^2(q-1)\frac{\varDelta^2 J^2}{[2]_q}}{\left(1 - q(q-1)\frac{\varDelta^2 J^2}{[2]_q}\right)_q^2} \times E_q\left[\frac{q}{[2]_q}(\varDelta J)^2\right]\varDelta^2 J\Big|_{J=0} = 0$$

where the second relation (11) is used. In similar manner, cumbersome but simple calculations give the field variance

$$\left\langle\phi^2\right\rangle_q^{(0)} = \varDelta^2 q. \tag{61}$$

[1] Generally speaking, a statistical ensemble might comprise of odd number of particles N that must arrive at unmatched factor in products (57). However, this factor is negligible within the thermodynamic limit $N \to \infty$.

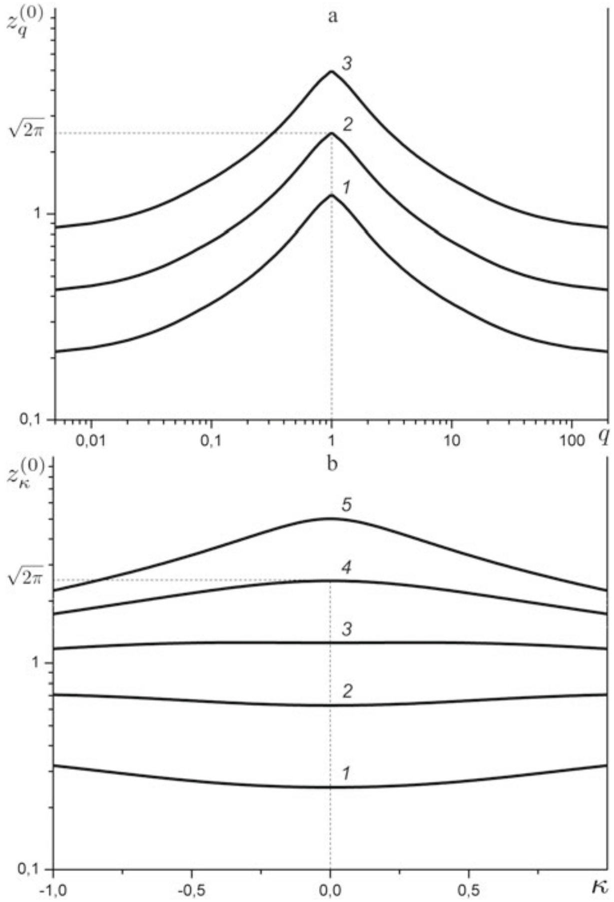

Fig. 1. Dependences of the one-site partition functions on the deformation parameters: (a) within the basic-deformed statistics at $\Delta = 0.5, 1, 2$ (curves 1, 2, 3, respectively); (b) within the Kaniadakis statistics (curves 1–5 correspond to $\Delta = 0.1, 0.25, 0.5, 1.0, 2.0$).

Thus, the basic-deformed distribution of free fields has zero moment of the first order and the variance, being proportional to the inverse curvature of the related action (56) and dependent on the deformation parameter linearly.

To develop a deformed perturbation theory one should, as usually, pick out an anharmonic contribution $V = V\{\phi\}$ in total action $S = S^{(0)} + V$ [19]. Then, the basic-deformed exponential can be written as follows:

$$e_q(-S) = E_q(-V)e_q\left(-S^{(0)}\right). \tag{62}$$

As a result, the formal expansion of the exponential $E_q(-V)$ in power series with consequent use of the differentiation rules (9) allows to present the generating functional (51) in the convenient form

$$\mathcal{Z}_q\{J\} = E_q\big(-V\{\mathcal{D}_J^q\}\big)\mathcal{Z}_q^{(0)}\{J\}. \tag{63}$$

Further, making use of the perturbation scheme with implementation of related diagram technique is straightforward [19]. Moreover, the thermodynamic limit $N \to \infty$ allows to use the Wick theorem to express higher correlators through the variance (61).

Similarly to the ordinary field scheme [19], an inconvenience of the above approach is that the generating functional (51) is non-additive value. To escape this drawback the Green functional

$$\mathcal{G}_q := \ln_q(\mathcal{Z}_q) \tag{64}$$

should be introduced to be deformed logarithm of the functional (51). It worthy to note that the function of the deformed logarithm has not an explicit form to be defined by the inverse exponential function $\mathcal{Z}_q = e_q(\mathcal{G}_q)$ given by the first series (1).

Since the Green functional (64) depends on an auxiliary field J, it can be more convenient to use a conjugate functional $\Gamma_q = \Gamma_q\{\phi\}$, which dependence of the initial field ϕ is provided by the Legendre transformation

$$\Gamma_q\{\phi\} := \sum_i J_i\phi_i - \mathcal{G}_q\{J\}. \tag{65}$$

The pair of the functionals $\mathcal{G}_q\{J\}$ and $\Gamma_q\{\phi\}$ plays the role of conjugated potentials, which basic-deformed variation yields the state equations

$$\phi_i = \mathcal{D}_{J_i}^q \mathcal{G}_q \quad \Leftrightarrow \quad J_i = \mathcal{D}_{\phi_i}^q \Gamma_q. \tag{66}$$

The first of these equations represents a generalization of the thermodynamic definition of the order parameter (for example, in the case of magnetic the magnetization equals derivative of the free energy over the magnetic field). The second equality (66) follows directly from the Legendre transformation (65) after its variation over the order parameter. Being analytical functional, conjugated potentials can be presented by the following series:

$$\mathcal{G}_q\{J\} = \sum_{n=1}^{\infty} \frac{1}{[n]_q!} \sum_{i_1\ldots i_n} \mathcal{G}_{i_1\ldots i_n}^{(n)} J_{i_1}\ldots J_{i_n}, \tag{67}$$

$$\Gamma_q\{\phi\} = \sum_{n=1}^{\infty} \frac{1}{[n]_q!} \sum_{i_1\ldots i_n} \Gamma_{i_1\ldots i_n}^{(n)} \eta_{i_1}\ldots\eta_{i_n}, \tag{68}$$
$$\eta_i \equiv \phi_i - \mathcal{G}_{i_1}^{(1)}.$$

To this end, related kernels $\mathcal{G}_{i_1\ldots i_n}^{(n)}$ and $\Gamma_{i_1\ldots i_n}^{(n)}$ reduce to n-particle Green function and its irreducible part respectively. Within the diagram representation, these kernels are depicted in Figure 2.

Following the standard field scheme [19], we show further that the generating functional (51) fulfils some formal relations. The first displays a system symmetry with respect to the basic-deformed variation in the form

$$\delta_q\phi_i = \epsilon_q f_i\{\phi\} \tag{69}$$

given by an analytical functional $f_i\{\phi\}$ in the limit $\epsilon_q \to 0$. Due to this variation the integrand of the last expression of

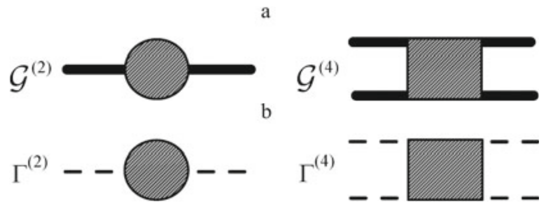

Fig. 2. Diagram representations of Green functions (a) and their irreducible parts (b).

the functional (51) is transformed in the following manner:

$$e_q\left[-S\{\phi+\delta_q\phi\}+J\cdot(\phi+\delta_q\phi)\right]\simeq e_q\left[(-S\{\phi\}+J\cdot\phi)\right.$$
$$\left.+\left(-\frac{\partial S}{\partial\phi_i}+J_i\right)\delta_q\phi_i\right]=e_q\Big(-S\{\phi\}+J\cdot\phi\Big)$$
$$\times E_q\left[\left(-\frac{\partial S}{\partial\phi_i}+J_i\right)\delta_q\phi_i\right]\simeq e_q\Big(-S\{\phi\}+J\cdot\phi\Big)$$
$$\times\left[1+\left(-\frac{\partial S}{\partial\phi_i}+J_i\right)\epsilon_q f_i\{\phi\}\right] \quad (70)$$

where the sum over repeated indexes is implied and the second expansion (1) is taken into account. On the other hand, the Jacobian determinant appearing due to passage from ϕ to $\phi+\delta_q\phi$ gives the factor $1+(\partial f_i/\partial\phi_i)\,\epsilon_q$. As a result, collecting multipliers, which include the infinitesimal value ϵ_q, we obtain from the invariance property of the generating functional (51):

$$\left[f_i\left\{\mathcal{D}_J^q\right\}\left(\frac{\partial S}{\partial\phi_i}\left\{\mathcal{D}_J^q\right\}-J_i\right)-\frac{\partial f_i}{\partial\phi_i}\left\{\mathcal{D}_J^q\right\}\right]\mathcal{Z}_q\{J\}=0. \quad (71)$$

Here we use the first of the state equations (66) for operator representations of the type $f_i\{\phi\}\,e_q\left(-S\{\phi\}+J\cdot\phi\right)=f_i\left\{\mathcal{D}_J^q\right\}e_q\left(-S\{\phi\}+J\cdot\phi\right)$. At condition $f_i\{\phi\}=\text{const}$, the equation (71) takes the simplified form, following directly from the generating functional (51) after variation over the field ϕ.

The second of above pointed equations allows to take into account arbitrary conditions $F_j\{\phi\}=0$, $j=1,2,\ldots$ for the set of fields $\{\phi_i\}$, $i=1,2,\ldots,N$ to be found. Consideration of these conditions are achieved by inserting the δ-functional $\delta\{F\}$ into the integrand of the expression (51) that results in introducing the prolonged form

$$\mathcal{Z}_q^{(F)}\{J\}:=\int e_q\left[-S\{\phi\}\right]E_q\left\{J\cdot\phi+\lambda\cdot F\right\}\{d_q\phi\}\{d_q\lambda\}. \quad (72)$$

Then variation over an auxiliary variables λ_j, $j=1,2,\ldots$ yields the desired result

$$F_i\left\{\mathcal{D}_J^q\right\}\mathcal{Z}_q^{(F)}\{J\}=0. \quad (73)$$

In comparison with the standard field theory [19], the essential peculiarity of the equalities (71) and (73) is that they contain the Jackson derivative $\mathcal{D}_J^q$ instead of the ordinary variation $\delta/\delta J$.

4 Finite-difference statistics

Within the framework of the finite-difference statistics, the field scheme is based on the definitions (20)–(23), (26) and (27) inherent in the h-calculus [48] to be developed in analogy with consideration stated in the previous section. With use of the h-exponential (20), the partition functional takes the form of the h-integral

$$Z_h\{\phi\}:=\int e_h\left[-\beta H(\mathbf{x})\right]\delta\left[\phi-\phi(\mathbf{x})\right]d_h\mathbf{x}$$
$$\equiv e_h\left(-S\{\phi\}\right) \quad (74)$$

instead of the expression (50). Respectively, the generating functional (51) is written as

$$Z_h\{J\}:=\int Z_h\{\phi\}e_h\{J\cdot\phi\}\{d_h\phi\}$$
$$=\int e_h\left(-S\{\phi\}+J\cdot\phi\right)\{d_h\phi\}. \quad (75)$$

Similarly to action of the Jackson derivative on the basic exponential, the differentiation operator (26) gives the correlator (53) with q substituted by h in the following form (cf. Eq. (55))

$$\langle\phi_{i_1}\ldots\phi_{i_n}\rangle_h=Z_h^{-1}\left(\mathcal{D}_{J_1}^h\ldots\mathcal{D}_{J_n}^h\right)Z_h\{J\}\big|_{J_1,\ldots,J_n=0}. \quad (76)$$

As in equation (56), the harmonic action

$$S^{(0)}\{\phi\}=\sum_i^N\frac{\phi_i^2}{[2]_h\Delta^2},\quad [2]_h=\frac{2h}{\ln(1+h)} \quad (77)$$

is determined with the inverse curvature Δ^2. Then, the generating functional (75) is expressed by the product

$$\mathcal{Z}_h^{(0)}\{J\}=\prod_i^N z_h^{(0)}(J_i) \quad (78)$$

of the type (58) with the one-particle factor (A.8). Taking into account the property (A.7) of the h-gamma function, one obtains the expression

$$z_0^{(0)}(J)=\sqrt{2\pi}\Delta e_h\left[\frac{[1]_h}{2}(\Delta J)^2\right],\quad [1]_h=\frac{h}{\ln(1+h)} \quad (79)$$

reducing to the standard form in the limit $h\to 0$. Moreover, the specific partition function

$$z_h^{(0)}=\int_{-\infty}^{+\infty}e_h\left(-\frac{\phi^2}{[2]_h\Delta^2}\right)d_h\phi \quad (80)$$

related to $J=0$ takes the non-deformed value $z_h^{(0)}=\sqrt{2\pi}\Delta$ for arbitrary parameters h.

According to the definition (76), the mean value of the h-deformed free fields equals

$$\langle\phi\rangle_h^{(0)}=[1]_h e_h\left[\frac{[1]_h}{2}(\Delta J)^2\right]\Delta^2 J\bigg|_{J=0}=0. \quad (81)$$

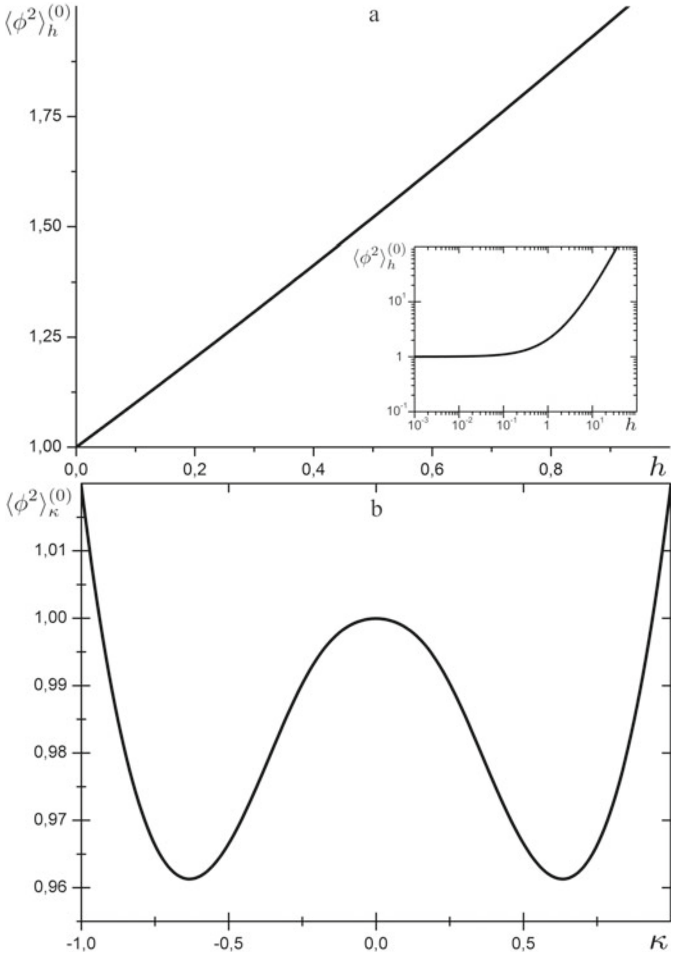

Fig. 3. Variances of free fields versus deformation parameters within the h-statistics (a) and the Kaniadakis one (b) at $\varDelta = 1$.

Respectively, the variance related is written as

$$\left\langle\phi^2\right\rangle_h^{(0)} = \varDelta^2[1]_h^2 \tag{82}$$

instead of equation (61). Thus, similarly to the basic-deformed distribution the h-deformed free fields have zero moment of the first order and the variance being proportional to the inverse curvature of the action (77) related. However, dependence of the variance (82) on the deformation parameter appears to be more strong than the linear dependence (61) related to the case of the basic deformation (see Fig. 3a).

The h-deformed perturbation theory is built in the complete accordance with the scheme stated in the previous section, with the only difference that the dual q-exponential $E_q(x)$ should be substituted with the self-dual h-exponential $e_h(x)$. Along this line, making use of the symbolic perturbation expansion of the type (62) yields the generating functional

$$\mathcal{Z}_h\{J\} = e_h\left(-V\left\{\mathcal{D}_J^h\right\}\right)\mathcal{Z}_h^{(0)}\{J\} \tag{83}$$

instead of equation (63). As well, application of the diagram technique and the Wick theorem appears to be straightforward. Moreover, following to the standard line [19], one should supplement the non-additive functional (75) by the Green functional $\mathcal{G}_h := \ln_h(\mathcal{Z}_h)$ where the h-logarithm is defined as (21) to be inverse to the h-exponential (20).

The functional $\varGamma_h = \varGamma_h\{\phi\}$ conjugated to the Green functional $\mathcal{G}_h\{J\}$ and the state equations related are defined by the Legendre transformation (65) and equations (66), respectively, where subscripts q are substituted by indexes h. With this substitution, the kernels of the series (67) and (68) are reduced to the n-particle Green function $\mathcal{G}^{(n)}_{i_1\ldots i_n}$ and its irreducible part $\varGamma^{(n)}_{i_1\ldots i_n}$ as depicted graphically in Figure 2. Finally, the formal relations for the generating functional (75) take the forms of equations (71), (72), and (73) to display a system symmetry with respect to the h-deformed variation $\delta_h\phi_i(x) = \phi_i(x+h) - \phi_i(x)$ and to take into account arbitrary conditions $F_j\{\phi\} = 0$, $j = 1, 2, \ldots$ for the set of fields $\{\phi_i\}$, $i = 1, 2, \ldots, N$ (in above pointed equations, q should be again substituted by h).

5 Kaniadakis statistics

Taking into account close likeness between basic-deformed, Tsallis and Kaniadakis statistics, we present the latter basing on the field-theoretical schemes developed in Section 3 and reference [17]. Doing so, the Tsallis deformed exponentials (28) should be substituted by the Kaniadakis ones (31) taking into account the multiplication rule (33) defined with the deformed sum (34). As a result, the generating functional (51) takes the form

$$\begin{aligned}\mathcal{Z}_\kappa\{J\} &:= \int \mathcal{Z}_\kappa\{\phi\}\exp_\kappa\{J\cdot\phi\}\{d_\kappa\phi\} \\ &= \int \exp_\kappa\left(-S\{\phi\} \overset{\kappa}{\oplus} J\cdot\phi\right)\{d_\kappa\phi\}\end{aligned} \tag{84}$$

where we use the shortening $J\cdot\phi = J_1\phi_1 \overset{\kappa}{\oplus} J_2\phi_2 \overset{\kappa}{\oplus} \cdots \overset{\kappa}{\oplus} J_N\phi_N$. According to the property (39), the correlator (cf. with Eq. (55))

$$\langle\phi_{i_1}\ldots\phi_{i_n}\rangle_\kappa = \mathcal{Z}_\kappa^{-1}\left(\mathcal{D}^{\kappa\phi_1}_{J_1}\ldots\mathcal{D}^{\kappa\phi_n}_{J_n}\right)\mathcal{Z}_\kappa\{J\}\Big|_{J_1,\ldots,J_n=0} \tag{85}$$

is determined by the partition function $\mathcal{Z}_\kappa = \mathcal{Z}_\kappa\{J=0\}$ and the Kaniadakis derivative (38). Then with use of the harmonic action

$$S^{(0)}\{\phi\} = \frac{1}{2\varDelta^2}\left(\phi_1^2 \overset{\kappa}{\oplus} \phi_2^2 \overset{\kappa}{\oplus} \cdots \overset{\kappa}{\oplus} \phi_N^2\right) \tag{86}$$

the generating functional (84) reduces to the product

$$\mathcal{Z}_\kappa^{(0)}\{J\} = \prod_i^N z_\kappa^{(0)}(J_i) \tag{87}$$

of the one-site factors

$$z_\kappa^{(0)}(J) = \int\limits_{-\infty}^{\infty} \exp_\kappa\left(-\frac{\phi^2}{2\varDelta^2}\right)\exp_\kappa(J\phi)\,d_\kappa\phi. \tag{88}$$

The specific partition function $z_\kappa^{(0)} = z_\kappa^{(0)}(J=0)$ takes the explicit form

$$z_\kappa^{(0)} = \int\limits_{-\infty}^{\infty} \left[\sqrt{1+\left(\frac{\kappa\phi^2}{2\Delta^2}\right)^2} - \frac{\kappa\phi^2}{2\Delta^2}\right]^{\frac{1}{\kappa}} \frac{\mathrm{d}\phi}{\sqrt{1+(\kappa\phi)^2}}. \tag{89}$$

According to equations (85) and (88), the first moment is $\langle\phi\rangle_\kappa^{(0)} = 0$, while the definition of the free field variance $\left\langle\phi^2\right\rangle_\kappa^{(0)}$ is achieved by inserting the ϕ^2 factor into the integrand of the integral (89) and dividing by $z_\kappa^{(0)}$.

As shows Figure 1b, dependence of the one-site partition function (89) on the deformation parameter κ has a symmetrical form with respect to the point $\kappa = 0$. Characteristically, the $|\kappa|$ arising results in the $z_\kappa^{(0)}$ increase at small values Δ^2 of the inverse curvature of action (86), while the Δ growth transforms the concave curve of the dependence $z_\kappa^{(0)}$ into the convex one. What about the dependence $\left\langle\phi^2\right\rangle_\kappa^{(0)}$ for the variance of the κ-deformed free fields, Figure 3b visualizes more complicated curve: first, the $|\kappa|$ growing results in the $\left\langle\phi^2\right\rangle_\kappa^{(0)}$ decrease from the value $\left\langle\phi^2\right\rangle_{\kappa=0}^{(0)} = \Delta^2$, after that the field variance increases before an anomalous value $\left\langle\phi^2\right\rangle_{|\kappa|=1}^{(0)} > \Delta^2$.

6 Generally deformed statistics

The examples considered in Sections 3–5 and reference [17] show the generalization of the deformed statistics should be performed along the two different lines. The first generalizes the basic-deformed and h-statistics, the second makes the same for statistics of the type proposed by Tsallis and Kaniadakis. Such a partition is stipulated by the principal difference between the multiplication rules (5) and (33). In the first case, this rule is provided by means of finding an exponential $\mathrm{E}_\lambda(x)$ being dual to the initial one $\mathrm{e}_\lambda(x)$ (this case is implemented for the basic- and h-calculi, and the latter relates to the self-dual exponential); the second class requires to deform the sum standing in the exponent of r.h.s. of Eq. (46) according to rules of the types (34) and (35). As was mentioned in the beginning of Section 2, these distinctions are caused by the principal difference between the basic and algebraic deformations, the first of which consists in a non-trivial deformation of the algebraic structure, whereas the second formalism keeps the algebraic structure in the standard form but introduces generalized operations. Let us consider the above mentioned cases separately.

Following the method developed in the end of Section 2, generalization of both basic-deformed and h-statistics is carried out in the straightforward manner. However, as comparison of the considerations stated in Sections 3 and 4 shows, to ensure the passage to the expression

$$\mathrm{Z}_\lambda\{J\} = \int \mathrm{e}_\lambda\left(-S\{\phi\} + J\cdot\phi\right)\{d_\lambda\phi\} \tag{90}$$

that contains the ordinary sum of exponents the deformed Laplace transform should be used

$$\mathrm{Z}_\lambda\{J\} := \int \mathrm{Z}_\lambda\{\phi\}\mathrm{E}_\lambda\{J\cdot\phi\}\{d_\lambda\phi\} \tag{91}$$

with the dual exponential $\mathrm{E}_\lambda(x) \Rightarrow E_q(x)$, being inherent in the basic calculus with $q=\lambda$, or the expression

$$\mathrm{Z}_\lambda\{J\} := \int \mathrm{Z}_\lambda\{\phi\}\mathrm{e}_\lambda\{J\cdot\phi\}\{d_\lambda\phi\} \tag{92}$$

with the initial exponential $\mathrm{e}_\lambda(x) \Rightarrow e_h(x)$, taking place in the h-deformed calculus with $h=\lambda$ (cf. Eqs. (51) and (75)). In all above cases, the n-fold derivative yields

$$\left(\mathrm{D}_{J_1}^\lambda \dots \mathrm{D}_{J_n}^\lambda\right)\mathrm{Z}_\lambda\{J\} = \int \left(\eta_\lambda(\phi_1)\dots\eta_\lambda(\phi_n)\right)\mathrm{e}_\lambda\left(-S\{\phi\} + J\cdot\phi\right)\{d_\lambda\phi\}. \tag{93}$$

Here, we take into account the differentiation rule

$$\mathrm{D}_{J_i}^\lambda \mathrm{e}_\lambda\left(-S+J\cdot\phi\right) = \eta_\lambda(\phi_i)\,\mathrm{e}_\lambda(-S+J\cdot\phi) \tag{94}$$

where an eigenvalue $\eta_\lambda(\phi_i)$ is determined by action of a generalized derivative $\mathrm{D}_{J_i}^\lambda$ with respect to a auxiliary field J_i (in the simple cases of both basic- and h-deformed calculi, one has $\eta_\lambda(\phi_i) = \phi_i$, while the Tsallis calculus relates to $\eta_\lambda(\phi_i) = \ln_{2-\lambda}(\phi_i)$ [17]). As a result, the correlator

$$\langle\eta_\lambda(\phi_1)\dots\eta_\lambda(\phi_n)\rangle_\lambda := \mathrm{Z}_\lambda^{-1}\int \left(\eta_\lambda(\phi_1)\dots\eta_\lambda(\phi_n)\right) \times \mathrm{Z}_\lambda\{\phi\}\{d_\lambda\phi\} \tag{95}$$

with the partition function

$$\mathrm{Z}_\lambda := \int \mathrm{Z}_\lambda\{\phi\}\{d_\lambda\phi\} = \int e_\lambda\left(-S\{\phi\}\right)\{d_\lambda\phi\} \tag{96}$$

is expressed in the form

$$\langle\eta_\lambda(\phi_1)\dots\eta_\lambda(\phi_n)\rangle_\lambda = \mathrm{Z}_\lambda^{-1}\left(\mathrm{D}_{J_1}^\lambda\dots\mathrm{D}_{J_n}^\lambda\right) \times\mathrm{Z}_\lambda\{J\}\big|_{J_1,\dots,J_n=0}. \tag{97}$$

Within the harmonic approach, the action (56) is written in the square-law form

$$S^{(0)} = \sum_{i=1}^{N}\frac{(\phi_i)_\lambda^2}{[2]_\lambda} \tag{98}$$

where a field set $\{\phi_i\}$ is distributed with the unit variance and λ-deformed square $(\phi_i)_\lambda^2$ and number $[2]_\lambda$ are used instead of equations (3) and (12). Then the generating functional (91) takes the form

$$\mathrm{Z}_\lambda^{(0)}\{J\} = \prod_{i'}^{[N/2]}\mathrm{z}_\lambda^{(0)}(J_{i'})\prod_{i''}^{[N/2]}\mathrm{z}_{1/\lambda}^{(0)}(J_{i''}) \tag{99}$$

of the type (58). Respectively, the product

$$\mathrm{Z}_\lambda^{(0)}\{J\} = \prod_i^N \mathrm{z}_\lambda^{(0)}(J_i) \tag{100}$$

is obtained for the generating functional (92). Here, each of multipliers related to one site is determined by the expression

$$\mathrm{z}_\lambda^{(0)}(J) = \sqrt{[2]_\lambda \pi_\lambda} \mathrm{E}_\lambda \left(\frac{J^2}{2\varDelta_\lambda^2} \right) \tag{101}$$

where $\varDelta_\lambda$ is a λ-deformed variance taking the value $\varDelta_{\lambda_0} = 1$ in the non-deformed limit $\lambda \to \lambda_0$; in turn, λ-deformed π-number is defined by equation of the type (A.5) with λ standing instead of q (as it stated above, a dual λ-exponential $\mathrm{E}_\lambda(x)$ should be used in the case of the type q-calculus and a self-dual λ-exponential $\mathrm{e}_\lambda(x)$ – for the class of h-type calculus). Consequently, the specific partition function $\mathrm{z}_\lambda^{(0)} = \mathrm{z}_\lambda^{(0)}(J=0)$ is given by the simple expression

$$\mathrm{z}_\lambda^{(0)} = \sqrt{[2]_\lambda \pi_\lambda}. \tag{102}$$

Similarly to the q- and h-statistical field theories, the mean value of free fields equals $\langle \eta(\phi) \rangle_\lambda^{(0)} \propto J|_{J=0} = 0$, while the variance $\left\langle \eta^2(\phi) \right\rangle_\lambda^{(0)}$ appears to be monotonically increasing function of the deformation parameter λ. Calculation of explicit form of dependences (101) and (102), as well as the variance $\left\langle \eta^2(\phi) \right\rangle_\lambda^{(0)}$ of order parameter $\eta = \eta(\phi)$ related to the derivation rule (94) needs to specify a concrete form of the λ-deformed exponentials.

The perturbation theory is based on the equation type of (62) and (63), as well as the diagram technique and the Wick theorem are built similarly to above considered field-theoretical schemes. Moreover, the additive generating functional related to the non-additive ones, (91) and (92), is expressed by the Green functional $\mathrm{G}_\lambda^{(0)}\{J\}$ in the form (64). The passage from this functional, dependent on an auxiliary field J, to the conjugate functional $\varGamma_\lambda = \varGamma_\lambda\{\phi\}$, being a functional of an order parameter ϕ, is achieved by the Legendre transformation type of (65), while the state equations have the form (66). Respectively, series of sort (67) and (68) have kernels that reduce to the n-particle Green function and its irreducible part. Within the diagram representation, these kernels look like depicted in Figure 2. Concerning the formal equations for the generating functionals (91) and (92) we should stress that they take the forms (71), (72), and (73) to display a system symmetry with respect to the λ-deformed variation $\delta_\lambda \phi_i(x)$ and take into account arbitrary conditions $F_j\{\phi\} = 0$, $j = 1, 2, \ldots$ for the set of fields $\{\phi_i\}$, $i = 1, 2, \ldots, N$. As it was pointed out above, in all above expressions the index q should be substituted by the subscript λ, and the λ-exponentials $\mathrm{e}_\lambda(x)$ and $\mathrm{E}_\lambda(x)$ should be used instead of the q-exponentials $e_q(x)$ and $E_q(x)$.

Finally, we state main principles of generalization of the algebraic deformed statistics. As it was stressed in the beginning of this Section, the cornerstone of calculi related is that the multiplication rules of the type (33) are defined by deformed sums of the type (34). As a result, the generating functional (84) takes the form

$$\begin{aligned} \mathrm{Z}_\lambda\{J\} &:= \int \mathrm{Z}_\lambda\{\phi\} \mathrm{e}_\lambda\{J \cdot \phi\}\{d_\lambda \phi\} \\ &= \int \mathrm{e}_\lambda \left(-S\{\phi\} \overset{\lambda}{\oplus} J \cdot \phi \right) \{d_\lambda \phi\} \end{aligned} \tag{103}$$

determined by a generally deformed sum standing in the last exponent. Respectively, the correlator (85) is written as follows:

$$\begin{aligned} \langle \eta_\lambda(\phi_{i_1}) \ldots \eta_\lambda(\phi_{i_n}) \rangle_\lambda = \mathrm{Z}_\lambda^{-1} \left(\mathrm{D}_{J_1}^{\lambda(\phi_1)} \ldots \mathrm{D}_{J_n}^{\lambda(\phi_n)} \right) \\ \times \mathrm{Z}_\lambda\{J\}|_{J_1,\ldots,J_n=0} \end{aligned} \tag{104}$$

with the partition function $\mathrm{Z}_\lambda = \mathrm{Z}_\lambda\{J=0\}$. Instead of equation (94), we take into account here the differentiation rule

$$\mathrm{D}_x^{\lambda(a)} \mathrm{e}_\lambda(ax) = \eta_\lambda(a) \mathrm{e}_\lambda(ax) \tag{105}$$

with an eigenvalue $\eta_\lambda(a)$ and a deformation $\lambda(a)$, being defined by action of a generalized derivative $\mathrm{D}_x^{\lambda(a)}$ with respect to a variable x at arbitrary constant a (for the Kaniadakis calculus, one has $\eta_\lambda(a) = a$ and $\lambda(a) = \lambda a$). Thereby, consideration of the simplest case of free fields requires to postulate an effective action as deformed sum of the type (86).

By analogy with above considered field-theoretical schemes, one finds partition function, perturbation theory, diagram technique, additive and conjugated Green functional, many-particle Green functions and their irreducible parts, as well as formal equations for the generating functional of systems that display a symmetry with respect to field variation and have some constraints.

7 Concluding remarks

Before discussion of the results obtained we should stress uppermost that quantum algebra and quantum groups, on which our approach is based, have been the subject of intense researches in different fields of the q-deformed quantum theory (see [33] and references therein). Moreover, use of the q-deformed algebra has allowed to develop multifractal theory [41,53] and thermostatistics of deformed bosons and fermions [28]. In our consideration, we restrict ourselves with generalization of the only classical thermostatistics, which version has been developed first in the work [11]. The principal peculiarity of the basic-deformed distribution arising in this model is to exhibit a cut-off in the energy spectrum which is generally expected in complex systems, which underlying dynamics is governed by long-range interactions.

In this connection it is worthy to note the deformed distribution proposed by Tsallis [1], which is characterized by the power-law asymptotic behaviour. The Tsallis picture is known to be inherent in self-similar statistical systems, which field theory has been built on the basis of both Mellin transform of the Tsallis exponential and Jackson derivative [17]. Contrary to the examples considered in Sections 3–6, a fluctuating order parameter of self-similar systems has non-zero mean value that reduces to the Tsallis deformed logarithm of the amplitude of a hydrodynamic mode. Formally, this is caused by using the Mellin transform with the power-law kernel $\phi^J = \exp[J\ln(\phi)]$ instead of the Fourier-Laplace one with the exponential kernel $\exp(J\phi)$.

In sections above we developed field-theoretical schemes based on both basic and algebraic deformations: the basic-deformed and finite-difference calculi relate to the first formalism, while the deformation procedures proposed by Tsallis, Abe, Kaniadakis, and Naudts are based on the second one. We construct generating functionals related and find their connection with corresponding correlators for basic-deformed, h-, and Kaniadakis calculi. Moreover, we introduce pair of additive functionals, which expansions into deformed series yield both Green functions and their irreducible proper vertices, as well as find formal equations, governing by the generating functionals of systems which possess a symmetry with respect to a field variation and are subjected to an arbitrary constrain. Finally, we generalize in the Naudts manner the field-theoretical schemes inherent in concrete calculi.

Concerning the physical results obtained above, we should point out peculiarities of dependences of both one-site partition function and variance of free fields on deformations (see Figs. 1 and 3, respectively). In the case of basic deformation, the specific partition function has in logarithmic axes symmetrical form with respect to the maximum point $q = 1$ (by this, increase of the inverse curvature Δ^2 of the effective action shifts up the curves of related dependences logarithmically equidistantly). For the h-deformation, the specific partition function takes non-deformed value. In the case of the Kaniadakis deformation, the dependence of the one-site partition function $z_\kappa^{(0)}$ on the deformation parameter κ has a symmetrical form with respect to the point $\kappa = 0$ (by this, the κ growth results in the $z_\kappa^{(0)}$ increase at small inverse curvature Δ^2, while the Δ growth transforms the concave curve of the dependence $z_\kappa^{(0)}$ into the convex one). Concerning the correlators of free-distributed fields, the only moment of the second order takes non-zero values. For all distributions, this moment is proportional to the inverse curvature Δ^2 to increase with the deformation parameter growth linearly in the case of the basic-deformed statistics and nonlinearly rapidly for the h-statistics. More complicated behaviour takes place for the Kaniadakis deformation, when the variance related decreases first and increases then up to an anomalous value.

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Appendix A

Within framework of the basic calculus, the pair of dual q-gamma functions is defined by the integrals

$$\gamma_q(\alpha) := \int_0^{+\infty} x_q^{\alpha-1} e_q(-x) d_q x, \tag{A.1}$$

$$\Gamma_q(\alpha) := \int_0^{+\infty} x_q^{\alpha-1} E_q(-qx) d_q x \tag{A.2}$$

where the upper limit equals $\frac{1}{1-q}$ for $|q| < 1$. These definitions are principle different from the introduced in reference [48] because we substitute the ordinary power function $x^{\alpha-1}$ by the deformed one $x_q^{\alpha-1}$ which generalizes the deformed binom (12) to substitute integer n by arbitrary exponent $\alpha - 1$. The definitions (A.1) and (A.2) ensure the properties

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha), \tag{A.3}$$
$$\gamma_q(\alpha+1) = [\alpha]_q \gamma_q(\alpha) q^{-\alpha} = -[-\alpha]_q \gamma_q(\alpha) \tag{A.4}$$

with $\gamma_q(0) = \gamma_q(1) = 1$ and $\Gamma_q(0) = \Gamma_q(1) = 1$. At $\alpha = 1/2$, the definition (A.1) gives the deformed Poisson integral

$$\int_{-\infty}^{+\infty} e_q\left(-x_q^2\right) d_q x = \frac{2}{[2]_q} \gamma_q(1/2) \equiv \sqrt{\pi_q} \tag{A.5}$$

where deformed π-number π_q has the limit $\pi_{q\to 1} = \pi \equiv 3.14159\ldots$

The specific generating functional is calculated as:

$$\begin{aligned} z_q^{(0)}(J) &= \int_{-\infty}^{+\infty} e_q\left(-\frac{\phi^2}{[2]_q\Delta^2}\right) E_q(J\phi) d_q\phi \\ &= E_q\left(q\frac{\Delta^2 J^2}{[2]_q}\right) \int_{-\infty}^{+\infty} e_q\left\{-\frac{\left(\phi-\Delta^2 J\right)_q^2}{[2]_q\Delta^2}\right\} d_q\phi \\ &= E_q\left(q\frac{\Delta^2 J^2}{[2]_q}\right) \int_{-\infty}^{+\infty} e_q\left(-\frac{x_q^2}{[2]_q\Delta^2}\right) d_q x \\ &= \frac{2\Delta}{\sqrt{[2]_q}} \gamma_q\left(\frac{1}{2}\right) E_q\left(q\frac{\Delta^2 J^2}{[2]_q}\right) \end{aligned} \tag{A.6}$$

where the variable $x = \phi - \Delta^2 J$ is introduced.

In the case of the h-deformed calculus, the gamma function related

$$\gamma_h(\alpha) := \int_0^{+\infty} x^{\alpha-1} e_h(-x) d_h x = [1]_h^{\alpha-1} \Gamma(\alpha) \tag{A.7}$$

appears to be proportional to the ordinary one $\Gamma(\alpha)$. By analogy with the calculations (A.6), the one-site generating functional is expressed as:

$$z_h^{(0)}(J) = 2\frac{[1]_h}{\sqrt{[2]_h}}\gamma_h\left(\frac{1}{2}\right)e_h\left[\frac{[1]_h}{2}(\Delta J)^2\right]\Delta$$
$$= \sqrt{[2]_h}\gamma_h\left(\frac{1}{2}\right)e_h\left[\frac{[1]_h}{2}(\Delta J)^2\right]\Delta. \quad \text{(A.8)}$$

Taking into account the property (A.7), the simple result (79) is obtained.

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